Virtual invariants of Quot schemes of surfaces

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joint work with
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Talk at FRAGMENT seminar

I. Hilbert scheme of points $S^{[n]}$

II. Two perspectives for $S^{[n]}$

III. Obstruction theory and virtual fundamental class

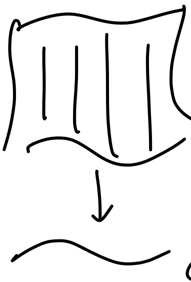
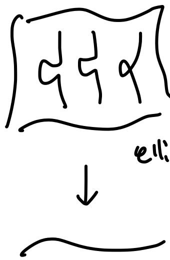
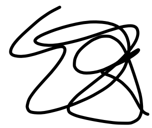
IV. Main results

V. Hilbert scheme of points & curves

VI. Sketch of proofs

I. Hilbert scheme of points $S^{[n]}$

S : smooth projective surface $/\mathbb{C}$

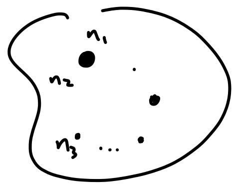
kod $= -\infty$	kod $= 0$	kod $= 1$	kod $= 2$
 ruled. C ⋮	• $\mathbb{C}^2/\Lambda$ • K3 surface ⋮	 elliptic C	 general type " $Zero(f) \subseteq \mathbb{P}^3$ " $\deg f \geq 5$

Question : How to compactify space of n distinct points of S ?

$$\left(S \times \cdots \times S \setminus \Delta\right) / S_n$$

1) topological : $\mathrm{Sym}^n S := (S \times \cdots \times S) / S_n$

↳ singular space



$\sum n_i = n$.

2) algebraic (analytic) : $S^{[n]}$ parametrizes length n subschemes.

e.g. $S^{[2]}$ parametrizes $\{p, q\}$ & $\{\overset{[v]}{p}\}$

$p \neq q$ $\qquad [v] \in (T_p S \setminus \{0\}) / \mathbb{C}^\times$

$\exists$ forgetful map : $S^{[n]} \longrightarrow \mathrm{Sym}^n S$

Thm (Fogarty) $S^{[n]}$ is connected and smooth of $2n$ dimension.

pf) Pick a point $[Z] \in S^{[n]}$ corresponding to

$$0 \to I_Z \to \mathcal{O}_S \to \mathcal{O}_Z \to 0.$$

By Grothendieck,

$$T_{[Z]} S^{[n]} = \mathrm{Hom}(I_Z, \mathcal{O}_Z).$$

Applying $\mathrm{Hom}(-, \mathcal{O}_Z)$, we obtain a long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{O}_S, \mathcal{O}_Z) \to \mathrm{Hom}(I_Z, \mathcal{O}_Z) \\ &\to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_S, \mathcal{O}_Z) \to \mathrm{Ext}^1(I_Z, \mathcal{O}_Z) \\ &\to \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^2(\mathcal{O}_S, \mathcal{O}_Z) \to \mathrm{Ext}^2(I_Z, \mathcal{O}_Z) \to 0 \end{aligned}$$

$\dim \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) = n$ & $\dim \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = \dim \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_S) = n$

$$\mathrm{ext}^{\bullet}(\mathcal{O}_Z, \mathcal{O}_Z) = \int \mathrm{ch}\,\mathcal{O}_Z . \mathrm{ch}\,\mathcal{O}_Z . \mathrm{td}(X) = 0$$

$$\| $$

$$n - \mathrm{ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) + n$$

$\therefore \dim T_{[Z]} S^{[n]} = 2n$ ▥

II Two perspectives for $S^{[n]}$

Recall that points in $S^{[n]}$ corresponds to

$$0 \to I_Z \to \mathcal{O}_S \to \mathcal{O}_Z \to 0 .$$

① ideal sheaf I_Z with $\det(I_Z) \simeq \mathcal{O}_S$

$\Downarrow$

"stable" sheaf with fixed determinant.

$\rightsquigarrow$

moduli of H-(semi)stable sheaves with fixed determinant

$$M_S^{H\text{-st}}(r, L, c_2) \underset{\text{open}}{\subseteq} \underbrace{M_S^{H\text{-ss}}(r, L, c_2)}_{\text{projective}} .$$

② quotient $\mathcal{O}_S \twoheadrightarrow \mathcal{O}_Z$

$\Downarrow$

quotient $\mathcal{O}_S^{\oplus N} \twoheadrightarrow Q$ with $\dim(\operatorname{supp} Q) \leq 1$.

$\rightsquigarrow$

Grothendieck's Quot scheme

$$\mathrm{Quot}_S(\mathbb{C}^N, \beta, n)$$

β: $c_1(Q)$ $\quad$ n: $\chi(Q)$

III. Obstruction theory and virtual fundamental class

S : surface

M : moduli of objects on S (e.g. moduli of sheaves, Quot schemes...)

Slogan : Study S via invariants of M,

$$\int_{[M]} (\text{tautological classes})$$

Difficulty : $\boxed{1}$ compactify moduli for well-defined integration.

$\boxed{2}$ Define a well-behaved class $[M] \in H_*(M, \mathbb{Z})$ or $A_*(M)$.

$\boxed{1}$ $M^{H\text{-ss}}$, Quot are projective.

$\boxed{2}$ Moduli spaces M often have an obstruction theory.

Roughly speaking, this means that M is locally modeled as

$$\begin{array}{c} V \\ \downarrow \Big) \, s : \text{section} \\ M \simeq \mathrm{Zero}(s) \hookrightarrow A \end{array}$$

where A is smooth, V : vector bundle.

$$\text{expected dim} = \dim(A) - \mathrm{rank}(V)$$

We say that an obstruction theory is "perfect" if

$$\text{expected dim} = \dim(\mathrm{Tan}_x) - \dim(\mathrm{Obs}_x)$$

is constant over each point $x \in M$.

Moduli of sheaves

Suppose that $M^{H\text{-st}} = M^{H\text{-ss}} \ni [E]$.

- $\mathrm{Tan}_{[E]} = \mathrm{Ext}^1(E,E)_0$
- $\mathrm{Obs}_{[E]} = \mathrm{Ext}^2(E,E)_0$

$\Big\}$ expected dim.

By stability, $\mathrm{Hom}(E,E)_0 = 0$ hence

$$\text{expected dim} = -\chi(E,E)_0 = \int \left(1 - \mathrm{ch}^\vee E \cdot \mathrm{ch} E\right) \cdot \mathrm{td}(T_S)$$

is constant over the moduli.

Quot scheme

Take $\mathrm{Quot}_S(\mathbb{C}^N, \beta, n) \ni x$ representing

$$0 \to S \to \mathcal{O}^{\oplus N} \to Q \to 0.$$

Then
- $\mathrm{Tan}_x = \mathrm{Hom}(S, Q)$
- $\mathrm{Obs}_x = \mathrm{Ext}^1(S, Q)$

$\Big\}$ expected dim.

Enough to notice that

$$\mathrm{Ext}^2(S,Q) \overset{SD}{\simeq} \mathrm{Hom}(Q, S\otimes K_S)^\vee = 0 .$$

torsion   torsion-free

$$\therefore \text{ exp. dim} = \chi(S,Q)$$
$$= \int ch^\vee S . ch Q . td(T_S)$$
$$= N\cdot n + \beta^2$$

By construction of [Behrend, Fantechi], [Li, Tian], we have

$$[M]^{vir} \in H_{2\cdot vd}(M, \mathbb{Z}) \text{ or } A_{vd}(M)$$

when M is either $M^{H\text{-st}} = M^{H\text{-ss}}$ or $\mathrm{Quot}_S(\mathbb{C}^N, \beta, n)$.

Using this, we define well-behaved invariants

$$\int_{[M]^{vir}} (\text{tautological classes}) .$$

<u>Advantage</u> : 1) Invariants of moduli spaces stay the same as we deform the underlying surface S.

2) $\exists$ various tools to study these invariants. Most notably, virtual localization [Graber, Pandharipande].

IV. Main results

① Tautological classes

Since Quot scheme is a fine moduli space, we have the universal quotient

$$0 \to \mathcal{S} \to \mathcal{O}^{\oplus N}_{\mathrm{Quot}\times S} \to \mathcal{Q} \to 0$$

$$\vdots$$

$$\begin{tikzcd} & \mathrm{Quot}\times S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathrm{Quot} & & S \end{tikzcd}$$

For each $\alpha \in K^0(S)$, we define

$$\alpha^{[n]} = \alpha^{[n]}_{N,\beta} := p_!\left(\mathcal{Q}\cdot q^*\alpha\right) \in K^0\left(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)\right).$$

② Virtual tangent bundle

$$T^{\mathrm{vir}}_{\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)} := R\mathcal{H}om_p(\mathcal{S},\mathcal{Q}) \in K^0\left(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)\right)$$

Given
- S : surface
- N, β
- $\alpha_1, \dots, \alpha_\ell \in K^0(S)$
- $k_1, \dots, k_\ell \in \mathbb{Z}_{\geq 0}$

define Homological / K-theoretic descendent series as

- $$Z^{H}_{S,N,\beta}(q \mid \underline{\alpha}, \underline{k}) := \sum_{n \in \mathbb{Z}} q^n \cdot \int_{[\mathrm{Quot}_S(\mathbb{C}^N, \beta, n)]^{\mathrm{vir}}} \prod_{i=1}^{\ell} c_{k_i}(\alpha_i^{[n]}) \cdot c(T^{\mathrm{vir}})$$

- $$Z^{K}_{S,N,\beta}(q \mid \underline{\alpha}, \underline{k}) := \sum_{n \in \mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}\left(\mathrm{Quot}_S(\mathbb{C}^N, \beta, n), \bigotimes_{i=1}^{\ell} \Lambda^{k_i} \alpha_i^{[n]} \otimes \Lambda_{-y} \Omega^{\mathrm{vir}}\right)$$

Since $\mathrm{vir.dim} = Nn + \beta^2 < 0$ for $n \ll 0$, these series are in

$$\mathbb{Q}((q)) \quad / \quad \mathbb{Q}(y)((q)).$$

Thm (AJLOP) If S is surface of $p_g > 0$, then

$$Z^{H}_{S,N,\beta}(q \mid \underline{\alpha}, \underline{k}) \in \mathbb{Q}(q) \quad \& \quad Z^{K}_{S,N,\beta}(q \mid \underline{\alpha}, \underline{k}) \in \mathbb{Q}(y)(q).$$

Conjecture : Rationality holds unconditionally.

When there are no tautological insertions, they specialize to generating series of $e^{\mathrm{vir}}(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n))$, $\chi^{\mathrm{vir}}_{-y}(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n))$:

$$Z^{e^{\mathrm{vir}}}_{S,N,\beta}(q) = \sum_{n\in\mathbb{Z}} q^n \cdot \int_{[\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)]^{\mathrm{vir}}} c(T^{\mathrm{vir}})$$

$$Z^{\chi^{\mathrm{vir}}_{-y}}_{S,N,\beta}(q) = \sum_{n\in\mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}\left(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n),\ \Lambda_{-y}\Omega^{\mathrm{vir}}\right).$$

These were motivated from Vafa–Witten invariants of S :

$$e^{\mathrm{vir}}(M^{H\text{-st}}_S(r,L,c_2)),\quad \chi^{\mathrm{vir}}_{-y}(M^{H\text{-st}}_S(r,L,c_2)).$$

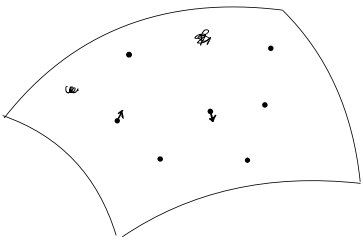
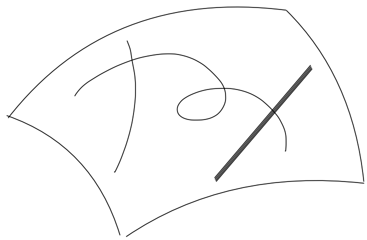
Opposed to rationality of Quot scheme invariants, they are expected to have certain modular property.

See [Göttsche, Kool] "Sheaves on surfaces and virtual invariants".

V. Hilbert scheme of points & curves

Philosophy: moduli of sheaves and quotients of S are built by

POINTS & CURVES

POINTS	CURVES
$S^{[n]}$	$\mathrm{Hilb}_S^\beta \quad (\beta \in H^2(S,\mathbb{Z}))$
	
zero-dimensional subscheme Z	effective divisor D
$0 \to I_Z \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$	$0 \to \mathcal{O}(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$
$\mathrm{Quot}_S(\mathbb{C}^1, \beta=0, n)$	$\mathrm{Quot}_S\left(\mathbb{C}^1, \beta, -\frac{\beta(\beta+K_S)}{2}\right)$
$\mathrm{vir.dim} = N\cdot n + \beta^2 = n \neq 2n$	$\mathrm{vd}_\beta = \frac{\beta(\beta-K_S)}{2}$

$$\begin{cases} \mathrm{Obs} = \mathrm{Ext}^1(I_Z, \mathcal{O}_Z) = H^0(K_S|_Z)^\vee : \text{rank } n \\ [S^{[n]}]^{vir} = e\left(K_S^{[n]\vee}\right) \cap [S^{[n]}] \end{cases}$$

Technique to study $\begin{cases} S^{[n]} & \rightsquigarrow \text{universality.} \\ \mathrm{Hilb}^{\beta} & \rightsquigarrow \text{Seiberg-Witten invariants.} \end{cases}$

① Universality of $S^{[n]}$

[Gottsche] $$\sum_{n=0}^{\infty} q^n \cdot \int_{S^{[n]}} c_{2n}(T_{S^{[n]}}) = \Big(\underbrace{\prod_{m=1}^{\infty} \frac{1}{1-q^m}}_{\text{universal series, i.e., independent of } S.} \Big)^{c_2(S)}.$$

(sketch of proof)

$\boxed{1}$ $$\int_{S^{[n]}} c_{2n}(T_{S^{[n]}}) = P_n(c_1(S)^2, c_2(S))$$

independent on S.

$\boxed{2}$ Multiplicativity : $$Z_S(q) := \sum_{n=0}^{\infty} q^n \cdot e(S^{[n]})$$

$$\Rightarrow Z_{S_1 \sqcup S_2}(q) = Z_{S_1}(q) \cdot Z_{S_2}(q).$$

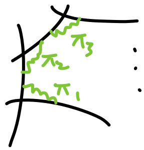
By $\boxed{1}$ + $\boxed{2}$, one can show that

$$\exists\ A(q), B(q) \in \mathbb{Q}[[q]]$$

s.t. $\quad Z_S(q) = A^{c_2(S)} \cdot B^{c_1(S)^2}$

$\boxed{3}$ Compute $Z_S(q)$ for special S.

S : toric surface

$(S^{[n]})^T = \{$  $\mid \sum_{i=1}^{\ell} |\pi_i| = n \}$.

$$e(S^{[n]}) = \#(S^{[n]})^T = \#\,(\pi_1, \dots, \pi_\ell) \text{ s.t. } \sum |\pi_i| = n.$$

$$\therefore\ Z_S(q) = \prod_{i=1}^{\ell} \sum_{n_i=0}^{\infty} p(n_i) \cdot q^{n_i} = \left(\frac{1}{\prod_{m=1}^{\infty} 1-q^m} \right)^{\ell}$$

$$\| $$

$$A^{c_2(S)} \cdot B^{c_1(S)^2}$$

Note that $\ell = c_2(S)$. $\Longrightarrow A = \dfrac{1}{\prod_{m=1}^{\infty} 1-q^m}$, $B = 1$ $\blacksquare$

② Seiberg-Witten invariants from Hilb_S^{β}

For simplicity of presentation, we assume $h^1(\mathcal{O}_S) = 0$.

$$\mathrm{Hilb}_S^{\beta} = |\mathcal{O}_S(D)| \simeq \mathbb{P}^{h^0(D)-1}$$

$$\therefore \quad [\mathrm{Hilb}_S^{\beta}]^{vir} = \underbrace{SW^{vd_\beta}(\beta)}_{\in \mathbb{Z}} \cdot [\mathbb{P}^{vd_\beta}] \in A_*(\mathbb{P}^{h^0(D)-1}).$$

One can compute it explicitly as

$$SW^{vd_\beta}(\beta) = \binom{h^1(D)-h^2(D)}{h^1(D)-h^2(D)+p_g}.$$

<u>Key property of SW invariants</u>

For surface S with $p_g > 0$, $SW^{vd_\beta}(\beta) \neq 0$

only if $\underline{vd_\beta = 0}$

in which case $SW^0(\beta) = \deg [\mathrm{Hilb}_S^{\beta}]^{vir}$.

e.g. S: minimal surface of general type with $p_g > 0$.

Only non-trivial SW invariants are

$$SW^0(0) = 1, \quad SW^0(K_S) = (-1)^{\chi(\mathcal{O}_S)}.$$

VI. Sketch of proofs

How does this philosophy work in practice for Quot schemes ?

Answer : Virtual localization + $[\mathrm{Quot}_S(\mathbb{C}^1,\beta,n)]^{vir}$ formula

Consider $T := \mathbb{C}^\times \curvearrowright \mathbb{C}^N$ with weights $w_1,\cdots,w_N$ $(w_i \neq w_j)$.

This induces $T \curvearrowright \mathrm{Quot}_S(\mathbb{C}^N,\beta,n)$ by

$$t\cdot\left[\mathcal{O}_S^{\oplus N} \xrightarrow{f} Q\right] := \left[\mathcal{O}_S^{\oplus N} \xrightarrow[\sim]{(t^{w_1},\cdots,t^{w_N})} \mathcal{O}_S^{\oplus N} \xrightarrow{f} Q\right].$$

Fixed loci decomposes to

$$\left(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)\right)^T = \coprod_{\substack{\beta=\beta_1+\cdots+\beta_N \\ n=n_1+\cdots+n_N}} \mathrm{Quot}_S(\mathbb{C}^1,\beta_1,n_1)\times\cdots\times\mathrm{Quot}_S(\mathbb{C}^1,\beta_N,n_N)$$

parametrizes

$$0 \to \bigoplus_{i=1}^N S_i[w_i] \to \bigoplus_{i=1}^N \mathcal{O}[w_i] \to \bigoplus_{i=1}^N Q_i[w_i] \to 0.$$

It is also easy to check that

$$\left[\left(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)\right)^T\right]^{vir} = \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ n=n_1+\cdots+n_N}} \left[\mathrm{Quot}_S(\mathbb{C}^1,\beta_1,n_1)\right]^{vir}\times\cdots\times\left[\mathrm{Quot}_S(\mathbb{C}^1,\beta_N,n_N)\right]^{vir}.$$

<u>Fact</u>: $\mathrm{Quot}_S(\mathbb{C}^1, \beta, n) \simeq S^{[m]} \times \mathrm{Hilb}_S^{\beta}$ where $m = n + \dfrac{\beta(\beta + K_S)}{2}$.

This is because for $N = 1$ case, we have

$$0 \longrightarrow \underset{\substack{\| \\ I_Z(-D)}}{S} \longrightarrow \mathcal{O} \longrightarrow Q \longrightarrow 0 .$$

What's even better is that $[\mathrm{Quot}_S(\mathbb{C}^1, \beta, n)]^{\mathrm{vir}}$ is explicit.

<u>Lemma</u> (L)

$$[\mathrm{Quot}_S(\mathbb{C}^1, \beta, n)]^{\mathrm{vir}} = e\Big(\overbrace{R\mathcal{H}om_p(\mathcal{O}_Z, \mathcal{O}(\mathcal{D}_\beta))}^{CO_\beta^{[m,0]}}\Big) \cap \Big([S^{[m]}] \times [\mathrm{Hilb}_S^{\beta}]^{\mathrm{vir}}\Big)$$

This uses the theory of nested Hilbert scheme by [Gholampour, Sheshmani, Yau], [Gholampour, Thomas].

$$\therefore \int_{[\mathrm{Quot}_S(\mathbb{C}^N, \beta, n)]^{\mathrm{vir}}} \tau_n = \sum_{\substack{\beta = \beta_1 + \cdots + \beta_N \\ n = n_1 + \cdots + n_N}} \int_{\prod_{i=1}^{N} [\mathrm{Quot}_S(\mathbb{C}^1, \beta_i, n_i)]^{\mathrm{vir}}} \frac{i^* \tau_n}{e(N^{\mathrm{vir}})}$$

$$= \sum_{\substack{\beta = \beta_1 + \cdots + \beta_N \\ n = n_1 + \cdots + n_N}} \int_{\prod_{i=1}^{N} [S^{[m_i]}] \times [\mathrm{Hilb}_S^{\beta_i}]^{\mathrm{vir}}} \prod_{i=1}^{N} e\left(CO_{\beta_i}^{[m_i, 0]}\right) \cdot \frac{i^* \tau_n}{e(N^{\mathrm{vir}})}$$

Suppose that $vd_{\beta_i} = \text{vir.dim}(\mathrm{Hilb}_S^{\beta_i}) = 0$ for all i.

(This is why we assume $p_g(S) > 0$)

Then we can separate out $\deg[\mathrm{Hilb}_S^{\beta_i}] \in \mathbb{Z}$ from integral:

$$\sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ n=n_1+\cdots+n_N}} SW(\beta_1)\cdots SW(\beta_N) \cdot \underbrace{\int_{\prod_{i=1}^N [S^{[m_i]}]} \prod_{i=1}^N e(\mathcal{O}_{\beta_i}^{[m_i,0]}) \cdot \frac{i^* \tau_n}{e(N^{vir})}}$$

Using the universality argument for $\star$, we obtain the structural formula for the following generating series:

$$\tilde{Z}^H_{S,N,\beta}(q,\underline{x},\underline{w}\,|\,\underline{\alpha}) := \sum_{n\in\mathbb{Z}} q^n \cdot \int_{[\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)]^{vir}} \prod_{i=1}^{\ell} c_{x_i}(\alpha_i^{[n]}) \cdot c(T^{vir})$$

Thm (L, AJLOP) S: surface of $p_g > 0$.

$$\tilde{Z}^H_{S,N,\beta}(q,\underline{x},\underline{w}\,|\,\underline{\alpha}) = q^{-\beta.K} \cdot A^{K^2} \cdot \prod_{s=1}^{\ell} B_s^{K.c_1(\alpha_s)}$$

$$\cdot \sum_{\beta=\beta_1+\cdots+\beta_N} SW(\beta_1)\cdots SW(\beta_N) \cdot \prod_{i=1}^N U_i^{\beta_i.K} \cdot \prod_{i,s} V_{is}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i<j} W_{ij}^{\beta_i.\beta_j}$$

Furthermore, all the universal series

$$A,\ B_s,\ U_i,\ V_{is},\ W_{ij} \in \mathbb{Q}(\underline{x})((w_1,\dots,w_N))[\![q]\!]$$

are known ~~explicitly~~ known up to change of variables

$$q = \prod_{s=1}^{\ell} \frac{1}{(1-x_s H)^{r_s}} \cdot \prod_{i=1}^{N} \frac{-H-w_i}{1-H-w_i} \longleftarrow$$

$H_1,\dots,H_N$ are solutions s.t. $H_i(q=0) = -w_i$

$$\begin{aligned}
A &= \prod_s \prod_{i=1}^{N} \frac{(1-x_s H_i)^{r_s}}{(1+x_s w_i)^{r_s}} \cdot \prod_{i,k\in[N]} \frac{H_i+w_k}{1+H_i+w_k} \cdot \prod_{\substack{i_1\neq i_2\\ i_1,i_2\in[N]}} \frac{1+H_{i_1}-H_{i_2}}{H_{i_1}-H_{i_2}} \\
&\quad \cdot \prod_{i=1}^{N}\left(\sum_s r_s \cdot \frac{x_s}{1-x_s H_i} + \sum_{k=1}^{N}\left(\frac{1}{1-H_i-w_k}+\frac{1}{H_i+w_k}\right)\right), \\
U_i &= \prod_s \frac{1}{(1-x_s H_i)^{r_s}} \cdot \prod_{k=1}^{N} \frac{1+H_i+w_k}{H_i+w_k} \cdot \prod_{\substack{i'\neq i\\ i'\in[N]}} \frac{H_{i'}-H_i}{1+H_{i'}-H_i} \cdot \frac{H_i-H_{i'}}{1+H_i-H_{i'}} \\
&\quad \cdot \left(\sum_s r_s \cdot \frac{x_s}{1-x_s H_i} + \sum_{k=1}^{N}\left(\frac{1}{1-H_i-w_k}+\frac{1}{H_i+w_k}\right)\right)^{-1}, \\
W_{ij} &= \frac{1+H_i-H_j}{H_i-H_j} \cdot \frac{1+H_j-H_i}{H_j-H_i}, \\
B_s &= \prod_{k=1}^{N} \frac{1+x_s w_k}{1-x_s H_k}, \\
V_{is} &= 1 - x_s H_i,
\end{aligned}$$

Rationality follows from

$$A,\ B_s,\ U_i,\ V_{is},\ W_{ij} \in \mathbb{Q}(\underline{x})((w_1,\dots,w_N))(H_1,\dots,H_N) \qquad //$$

How do we obtain these universal series explicitly?

<u>Answer</u> : "cosection localization" [Kiem, Li]

Suppose that $\exists$ smooth canonical divisor $C = \mathrm{Zero}(\omega) \in |K_S|$.

(This is okay because we can choose special geometry for the computation of universal series.)

Recall that for $x = [0 \to S \to \mathcal{O}^{\oplus N} \to Q \to 0] \in \mathrm{Quot}_S(\mathbb{C}^N, \beta, n)$,

$$\mathrm{Tan} = \mathrm{Hom}(S, Q), \quad \mathrm{Obs} = \mathrm{Ext}^1(S, Q).$$

Using a canonical curve C, we define a cosection

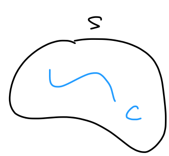
$$\theta : \mathrm{Ext}^1(S,Q) \longrightarrow \mathrm{Ext}^2(Q,Q) \xrightarrow{\text{trace}} H^2(\mathcal{O}_S) \xrightarrow{\int_S \omega \wedge -} \mathbb{C}.$$

This gives a cosection of the obstruction sheaf

$$\begin{tikzcd} \mathrm{Obs} \arrow[r, "\theta"] \arrow[d] & \mathcal{O}_{\mathrm{Quot}} \\ \mathrm{Quot}_S(\mathbb{C}^N, \beta, n) & \end{tikzcd}$$

s.t. $\mathrm{Zero}(\theta) = \mathrm{Quot}_C(\mathbb{C}^N, \beta, n) \subseteq \mathrm{Quot}_S(\mathbb{C}^N, \beta, n)$

locus where the quotient Q has $\mathrm{supp}(Q) \subseteq C$.



This suggest that the calculation may be done over the Quot scheme over a canonical curve C.

This can be done explicitly using combinatorics.

e.g. $C = \mathbb{P}^1 \rightsquigarrow C^{[n]} = \mathbb{P}^n$.

$$\sum_{n=0}^{\infty} q^n \int_{C^{[n]}} c\left(\mathcal{O}(d)^{[n]}\right) = ?$$

1) $\mathcal{O}(d)^{[n]} = \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n-d-1)} - \mathcal{O}_{\mathbb{P}^n}^{\oplus(d+1)} \in K^0(\mathbb{P}^n)$

2) $c\left(\mathcal{O}(d)\right)^{[n]} = (1-h)^{n-d-1}$

3) $$\sum_{n=0}^{\infty} q^n \int_{\mathbb{P}^n} c\left(\mathcal{O}(d)^{[n]}\right) = \sum_{n=0}^{\infty} q^n \cdot [h^n]\, (1-h)^n \cdot (1-h)^{-d-1}$$

$$= \sum_{n=0}^{\infty} q^n\, [h^n]\, \Phi(h)^n \cdot \psi(h)$$

$$= \frac{\psi(h)}{K(h)} \qquad \text{(Lagrang-Bürmann)}$$

where
$$\begin{cases} K(h) = 1 - \dfrac{h \cdot \Phi'(h)}{\Phi(h)} = \dfrac{1}{1-h} \\ q = \dfrac{h}{\Phi(h)} = \dfrac{h}{1-h} \end{cases}$$

$$\therefore \sum_{n=0}^{\infty} q^n \int_{\mathbb{P}^n} c\left(\mathcal{O}(d)^{[n]}\right) = \frac{(1-h)^{-d-1}}{(1-h)^{-1}} = \frac{1}{(1-h)^d} = (1+q)^d .$$

where $q = \dfrac{h}{1-h} \iff h = \dfrac{q}{1+q}$

Of course, this simple example can be done without Lagrange–Bürmann.

This method has also been applied to study virtual Segre / Verlinde series of Quot schemes:

$$\begin{cases} S_{s,N}(q\,|\,\alpha) := \displaystyle\sum_{n\in\mathbb{Z}} q^n \cdot \int_{[\mathrm{Quot}_s(\mathbb{C}^N,n)]^{\mathrm{vir}}} s\left(\alpha^{[n]}\right) \\ V_{s,N}(q\,|\,\alpha) := \displaystyle\sum_{n\in\mathbb{Z}} q^n \cdot \chi^{\mathrm{vir}}\left(\mathrm{Quot}_s(\mathbb{C}^N,n),\, \det \alpha^{[n]}\right) . \end{cases}$$

We have a surprisingly simple Segre / Verlinde correspondence.

Thm (AJLOP) $S_{s,N}((-1)^N q\,|\,\alpha) = V_{s,N}(q\,|\,\alpha)$.

We also exhibit symmetry exchanging $N \longleftrightarrow rk(\alpha)$.

Define $\mu(\alpha) := \dfrac{k \cdot c_1(\alpha)}{rk(\alpha)}$.

Thm (AJLOP) • $\alpha, \tilde{\alpha}$: K-theory classes of rank $r, N \geq 1$

• $\mu(\alpha) = \mu(\tilde{\alpha})$.

$\Rightarrow$ $V_{s,N}(q|\alpha) = V_{s,r}(q|\tilde{\alpha})$, in other words,

$$\chi^{vir}(\mathrm{Quot}_s(\mathbb{C}^N, n), \det \alpha^{[n]}) = \chi^{vir}(\mathrm{Quot}_s(\mathbb{C}^r, n), \det \tilde{\alpha}^{[n]})$$

This reminds the level-rank duality (strange duality) for moduli of stable bundles of curves.

Thank You